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On the finite-temperature generalization of the C-theorem and the interplay between classical and quantum fluctuations

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Abstract. The behaviour of the finite-temperature C-function, defined by Neto and Fradkin (1993 *Nucl. Phys.* B **400** 525), is analysed within a d-dimensional exactly solvable lattice model, recently considered by Vojta (1996 *Phys. Rev.* B **53** 710), which is of the same universality class as the quantum nonlinear O(n) sigma model in the limit $n \to \infty$. The scaling functions of C for the cases $d = 1$ (absence of long-range order), $d = 2$ (existence of a quantum critical point), $d = 4$ (existence of a line of finite-temperature critical points that ends up with a quantum critical point) are derived and analysed. The locations of regions where C is monotonically increasing (which depend significantly on d) are exactly determined. The results are interpreted within the finite-size scaling theory that has to be modified for $d = 4$.

1. Introduction

The original Zamolodchikov C-theorem is related to zero-temperature quantum systems. It establishes the existence of a dimensionless function C of the coupling constants with monotonic properties along the renormalization group trajectories [1]. The assumptions presented in the proof are related with the energy–momentum conservation, the rotational and translational symmetries, and positivity in a two-dimensional (2D) quantum field theory. The behaviour of the C-function reflects the role of the quantum fluctuations and it is useful in determining the qualitative features of the theory away from the criticality. At the fixed points it takes the value of the central charge of the corresponding conformal field theory. Since the basic assumptions underlying the C-theorem are not only specific to two dimensions, considerable interest exists in generalization of the Zamolodchikov result for dimensionalities different from two as well as for nonzero temperatures [2–6]. Earlier efforts (see [2] and references therein) have been devoted to finding a version of the C-theorem valid in four dimensions. There the approach was based on a careful investigation of the form of the trace of the energy–momentum tensor, written in terms of finite local composite operators. Despite being able to write expressions for the Zamolodchikov equations for the C-function, similar to the case of two dimensions, it turns out that it is not possible to demonstrate the monotonicity property. Let us note also the fact that the 3D analogue of central charge [7] is not equivalent to the universal number characterizing the size dependence of the free energy at the critical point [4] (which is always the case in 2D conformal field theory). This fact indicates that a straightforward generalization of the Zamolodchikov C-theorem is not to be expected for a general d. See also [5], where different approaches to the problem have been offered and where

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it was shown that no direct relations exist between the 'finite-temperature C-theorem' and the Zamolodchikov C-theorem at zero temperature. In the present study an approach, proposed by Netto and Fradkin [3] (in some sense a thermodynamic one), for finding a candidate for the C-function will be considered. In [3] the following dimensionless function is defined:

$$C(\beta, g, a|d) = -\beta^{d+1}\frac{v^d}{n(d)}[f(\beta, g, a|d) - f(\infty, g, a|d)] \tag{1.1}$$

where β is the inverse temperature ($\beta = 1/T$) with the Boltzman constant $k_B = 1$, g is a set of dimensionful coupling constants, $f(\infty, g, a|d) \equiv E_0(g, a|d)$ is the zero-temperature energy density, i.e. the energy of the 'infinite' in the inverse temperature system, $f(\beta, g, a|d)$ is the full free-energy density (per unit volume) of the system, and a is the characteristic length scale of the lattice. Here $n(d)$ is a positive real number (which depends only on the dimensionality d of the system) and v is the characteristic velocity (e.g. the velocity of quasiparticles) in the system. The function $C(\beta, g, a|d)$ is considered to be the d-dimensional nonzero temperature extension of the Zamolodchikov C-function. It is supposed to be *positive*, and, in the regions where the quantum fluctuations dominate, a *monotonically increasing function of the temperature*. In [3] the numbers $n(d) = \Gamma((d+1)/2)\zeta(d+1)/\pi^{(d+1)/2}$ for bosons ($\zeta(x)$ is the Riemann zeta function, $\Gamma(x)$ is the gamma function) and $n(d) = \Gamma((d+1)/2)\zeta(d+1)(2-2^{1-d})/\pi^{(d+1)/2}$ for fermions have been suggested. Obviously, the exact choice of $n(d)$ does not effect the monotonicity properties of the C-function. Functions analogous to the one defined in (1.1) have also been discussed in [4–6].

For a general d the existence of phase transitions in the system, as well as the interplay between the quantum and classical fluctuations, makes the analysis of the behaviour of the C-function difficult from a general point of view. That is why, for any $d \neq 1$ the properties of C have been considered on the examples of concrete models: the free massive field theories (for any d) [3], the Ising model in a transverse field (for $d = 1$) [3] and the quantum nonlinear sigma model (QNLσM) in the limit $N \to \infty$ and $d = 2$ [3, 4]. (Recently, the value of the C-function at the critical point as a function of d , $1 < d < 3$, has been calculated in [6] for that model.)

In the present paper we will consider the d-dependence of the monotonicity property of the C-function within the framework of an exactly solvable lattice model. We will explicitly demonstrate the crucial role that the existence of a quantum ($T = 0$) and/or classical ($T \neq 0$) critical points plays for the behaviour of C in different regions of the phase diagram in the plane temperature—the parameter controlling the quantum fluctuations.

The paper is organized as follows. In section 2 we briefly describe the model and in section 3 present the basic exact analytical expressions for the free energy of the bulk system at nonzero temperature. Section 4 contains the analysis of the behaviour of the C-function in dimensions one, two and four in different regimes of the parametric space of the model. Section 5 presents a finite-size scaling (FSS) interpretation of the results. The paper closes with concluding remarks given in section 6.

2. The model

The model we consider describes a magnetic ordering due to the interaction of quantum spins. It dates back to the work of G Obermair (1972) [8], in which a canonical quantization for a dynamical version of the spherical model was proposed. Later Srednicki [9], using the Feynman-path integral formalism, extracted from a 2D mean spherical model a quantum mechanical Hamiltonian which is a 1D version of the model proposed by Obermair. Henkel and Hoeger [10] generalized that result to d dimensions. Recently, [11] has renewed the interest in this scheme in the context of quantum phase transitions at zero temperature. The Hamiltonian has the following form [11]:

$$\mathcal{H} = \frac{1}{2} g \sum_{\ell} \mathcal{P}_{\ell}^2 - \frac{1}{2} \sum_{\ell\ell'} J_{\ell\ell'} \mathcal{S}_{\ell} \mathcal{S}_{\ell'} + \frac{\mu}{2} \sum_{\ell} \mathcal{S}_{\ell}^2 \tag{2.1}$$

where $\mathcal{S}_\ell$ are spin operators at site ℓ, the operators $\mathcal{P}_\ell$ are 'conjugated' momenta (i.e. $[\mathcal{S}_\ell, \mathcal{S}_{\ell'}] = 0, [\mathcal{P}_\ell, \mathcal{P}_{\ell'}] = 0$, and $[\mathcal{P}_\ell, \mathcal{S}_{\ell'}] = \mathrm{i}\delta_{\ell\ell'}$, with $\hbar = 1$), the coupling constants $J_{\ell,\ell'} = J$ are between nearest neighbours only, the coupling constant g is introduced so as to measure the strength of the quantum fluctuations (below it will be called quantum parameter), and, finally, the spherical field μ is introduced to ensure the fulfilment of the constraint $\langle \sum_\ell \mathcal{S}_\ell^2 \rangle = N$. Here $\langle \cdots \rangle$ denotes the standard thermodynamic average taken with $\mathcal{H}$.

In the thermodynamic limit the reduced free energy $\tilde{f}_\infty(\beta, g|d) = f_\infty(\beta, g|d)/\sqrt{gJ}$ takes the form†

$$\lambda \tilde{f}_\infty(t, \lambda|d) = \sup_{\phi} \left\{ \frac{t}{(2\pi)^d} \int_{-\pi}^{\pi} \mathrm{d}q_1 \ldots \int_{-\pi}^{\pi} \mathrm{d}q_d \right. \\ \left. \times \ln \left[2 \sinh \left(\frac{\lambda}{2t} \sqrt{\phi + 2 \sum_{i=1}^{d} (1 - \cos q_i)} \right) \right] - \frac{1}{2}\phi \right\} - d \tag{2.2}$$

where we have introduced the notations: $\lambda = \sqrt{g/J}$ is the normalized quantum parameter, $t = \frac{T}{J}$ is the normalized temperature and $\phi = \frac{\mu}{J} - 2d$ is the shifted spherical field. The supremum is attained at a solution of the mean-spherical constraint that reads

$$1 = \frac{t}{(2\pi)^d} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \mathrm{d}q_1 \ldots \int_{-\pi}^{\pi} \mathrm{d}q_d \frac{1}{\phi + 2\sum_{i=1}^{d}(1 - \cos q_i) + b^2 m^2} \tag{2.3}$$

where $b = 2\pi t/\lambda$.

Equations (2.2) and (2.3) provide the basis for studying the critical behaviour of the model under consideration.

The critical behaviour and some finite-size properties of this model have been considered in [12, 13] for $1 < d < 3$. Below we present a brief sketch of the derivation of the bulk free energy for $d = 1, 2, 4$ at low temperatures.

3. The free energy at low temperatures

By using the identities

$$\ln \frac{\sinh b}{\sinh a} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \ln \frac{b^2 + \pi^2 m^2}{a^2 + \pi^2 m^2} \tag{3.1}$$

where $ab > 0$, a, b are arbitrary real numbers,

$$\ln(a + b) = \ln a + \int_0^\infty \exp(-ax)(1 - \exp(-bx)) \frac{\mathrm{d}x}{x} \tag{3.2}$$

where $a > 0$, $a+b > 0$, and the Jacobi identity after some algebra at low temperatures ($\frac{\lambda}{t} \gg 1$) the expression for the free energy (2.2) can be rewritten in the form

$$2\lambda \tilde{f}_\infty(t, \lambda|d) = \lambda a(\phi, d) - (\phi + 2d) - \lambda s(\phi, b, d) \tag{3.3}$$

† The lattice constant is taken to be $a = 1$ and the dependence on it will be omitted hereafter.

where

$$a(\phi, d) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{\mathrm{d}x}{x^{3/2}} \exp(-x\phi)[1 - (\exp(-2x) I_0(2x))^d] + \sqrt{\phi} \tag{3.4}$$

$$s(\phi, b, d) = 2 \int_0^\infty \frac{\mathrm{d}x}{x} (4\pi x)^{-(d+1)/2} \exp(-x\phi) R\left(\frac{\pi^2}{xb^2}\right) \tag{3.5}$$

$$R(x) = \sum_{m=1}^{\infty} \exp(-xm^2). \tag{3.6}$$

$I_0(x)$ is a modified Bessel function, and ϕ in (3.3) is the solution of the corresponding spherical field equation

$$\frac{\partial}{\partial \phi}[\lambda a(\phi, d) - (\phi + 2d) - \lambda s(\phi, b, d)] = 0. \tag{3.7}$$

The above expressions are valid for *any* d.

In the remainder we will consider the dimensions $d = 1$, $d = 2$ and $d = 4$.

(a) For $d = 1$

$$a(\phi, 1) = \frac{1}{2} {}_2F_1\left(-\frac{1}{4}, \frac{1}{4}, 1, \frac{4}{(2+\phi)^2}\right) \sqrt{2+\phi} \tag{3.8}$$

$$s(\phi, b, 1) = -\frac{b}{2\pi^2} \sqrt{\phi} \sum_{m=1}^{\infty} m^{-1} K_1\left(\frac{2\pi m \sqrt{\phi}}{b}\right) \tag{3.9}$$

where ${}_2F_1$is the hypergeometric function and $K_1(x)$ is the MacDonald function (second modified Bessel function).

(b) For $d = 2$, and $\phi \ll 1$

$$a(\phi, 2) \simeq a(0, 2) + \mathcal{W}_2(0)\phi - \frac{1}{6\pi}\phi^{3/2} \tag{3.10}$$

$$s(\phi, b, 2) = -\left(\frac{b}{2\pi}\right)^3 \left[\frac{\sqrt{\phi}}{b} \mathrm{Li}_2\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right) + \frac{1}{2\pi} \mathrm{Li}_3\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right)\right] \tag{3.11}$$

where

$$\mathcal{W}_d(\phi) = \frac{1}{2(2\pi)^d} \int_{-\pi}^{\pi} \mathrm{d}q_1 \ldots \int_{-\pi}^{\pi} \mathrm{d}q_d \left(\phi + 2\sum_{i=1}^{d}(1 - \cos q_i)\right)^{-1/2} \tag{3.12}$$

is a Watson-type integral, $\mathcal{W}_2(0) \approx 0.3214$, and $\mathrm{Li}_n(x)$ is the polylogarithmic function.

(c) For $d = 4$, and $\phi \ll 1$

$$a(\phi, 4) \simeq a(0, 4) + \mathcal{W}_4(0)\phi - \frac{1}{2} r\phi^2 + \frac{1}{30\pi^{3/2}}\phi^{5/2} \tag{3.13}$$

$$\begin{aligned} s(\phi, b, 4) = -\left(\frac{b}{2\pi}\right)^5 \bigg[&\frac{\phi}{b^2} \mathrm{Li}_3\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right) \\ &+ \frac{3\sqrt{\phi}}{2\pi b} \mathrm{Li}_4\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right) + \frac{3}{4\pi^2} \mathrm{Li}_5\left(\exp\left(-\frac{2\pi\sqrt{\phi}}{b}\right)\right)\bigg] \end{aligned} \tag{3.14}$$

where $\mathcal{W}_4(0) \approx 0.1891$ and

$$r = \int_0^\infty \sqrt{x}[\exp(-2x) I_0(2x)]^4 \, \mathrm{d}x \approx 0.0677. \tag{3.15}$$

Now we have the basic expressions needed to analyse the behaviour of the finite-temperature C-function as defined by equation (1.1).

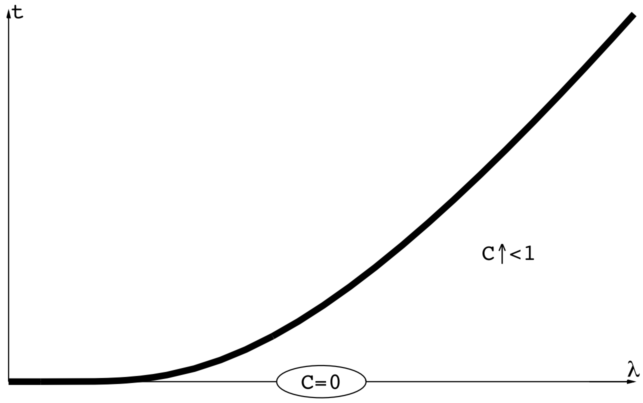

Figure 1. The bold curve $t_{LT} = 8\lambda \exp(-2\pi/\lambda)$ borders from above the region in the t–λ plane where the expression for the 1D C-function, given by equation (4.1), is valid. The symbol $C\uparrow$ means that C increases in the whole of that region starting from $C = 0$ at $t = 0$.

4. The behaviour of the C-function

4.1. The case $d = 1$

From equations (3.3), (3.8) and (3.9) it is easy to see that the only nonanalyticity in the behaviour of the free energy exists at $\phi = 0$. Then, for $0 < \phi \ll 1$ one obtains from equation (1.1), after some algebra and identifying $v = \sqrt{gJ}$, that the C-function† can be written in the following scaling form:

$$C(t, \lambda) = \frac{\sqrt{\pi/2}}{6} y_0^{1/4} \exp(-\sqrt{y_0}) \tag{4.1}$$

where the scaling variable is $y_0 = \lambda^2 \phi_0 / t^2$. Here

$$\phi_0 = 64 \exp(-4\pi/\lambda). \tag{4.2}$$

The solution ϕ_0 of the corresponding spherical field equation for the zero-temperature system has an essential singularity at $\lambda = 0$ (see also [9]). Such type of solution is very well known from different problems, e.g. one-dimensional anharmonic crystal [14] and the quantum nonlinear O(N) sigma model in the large N limit (see, e.g., [15, 16]). In deriving (4.1) we have been interested in such a behaviour of the nonzero temperature system which approaches the corresponding zero-temperature behaviour when $T \to 0$.

As is clear from the above expressions, C is a positive and monotonically increasing function of the temperature.

The behaviour of the C-function for $d = 1$ is illustrated in figure 1.

4.2. The case $d = 2$

We are interested in the behaviour of the C-function around and below the critical point only, i.e. $0 < \phi \ll 1$ [12,13]. As is well known, the critical point is at $\lambda = \lambda_c = 1/\mathcal{W}_2(0) \approx 3.1114$ and $T = 0$. Then, taking into account in equation (1.1) that $n(2) = \zeta(3)/(2\pi)$ and $v = \sqrt{gJ}$,

† For simplicity of notation the dependence on the argument d of the C-function is omitted hereafter.

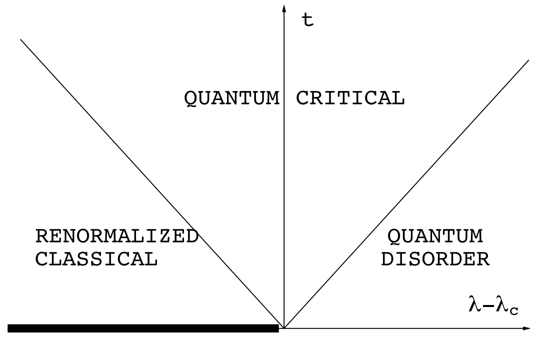

Figure 2. The phase diagram of the model and crossovers for the case $d = 2$ as a function of t and the normalized quantum parameter λ. One distinguishes renormalized classical, quantum critical and quantum disordered regions. Long-range order is present only at $t = 0$ for $\lambda < \lambda_c$.

for the C-function we obtain from equations (3.10) and (3.11) that $C(t, \lambda) = X(x)$, where

$$X(x) = \frac{1}{\zeta(3)}\left[x(y - y_0) + \frac{1}{6}(y^{3/2} - y_0^{3/2}) + \sqrt{y}\mathrm{Li}_2(\exp(-\sqrt{y})) + \mathrm{Li}_3(\exp(-\sqrt{y}))\right] \quad (4.3)$$

with $x = \pi(1/\lambda - 1/\lambda_c)\lambda/t$. Here $y = y(x)$, and $y_0 = y_0(x)$ are solutions of the corresponding equations that follow from (4.3) by requiring the first partial derivative of the rhs of (4.3) with respect to y, and y_0, respectively, to be zero. These solutions are

$$\sqrt{y} = 2\mathrm{arcsh}(\tfrac{1}{2}\exp(-2x)) \quad (4.4)$$

and

$$\sqrt{y_0} = \begin{cases} -4x & \lambda > \lambda_c \\ 0 & \lambda \leqslant \lambda_c. \end{cases} \quad (4.5)$$

Equation (4.3) determines the *exact scaling function of C* for the case $d = 2$. From the above equations one can see the different behaviour of y in the three regions: (i) *renormalized classical*, where y tends to zero exponentially fast as a function of x ($x \gg 1$); (ii) *quantum critical*, where $y = \mathrm{O}(1)$ (for $x = \mathrm{O}(1)$); (iii) *quantum disordered*, where y diverges as $(4x)^2$ for $x \ll -1$; ($y \sim (\chi t^2)^{-1}$, where χ is the susceptibility of the system, see [12]). The location of these regions is depicted in figure 2.

The behaviour of the C-function reflects the existence of these three regions.

When $\lambda < \lambda_c$ and $t \to 0$, from equations (4.3)–(4.5) it follows that

$$C(t, \lambda) \simeq 1 - \frac{1}{4\zeta(3)}\exp[-4\pi(1 - \lambda/\lambda_c)t^{-1}]. \quad (4.6)$$

One explicitly observes the exponentially small corrections to the limit value of $C = 1$ (at $t = 0$) that corresponds to massless bosons in d dimensions [3, 4].

At $\lambda = \lambda_c$, $C(t, \lambda)$ simplifies, by using the Sachdev identity [4], and becomes

$$C(t, \lambda_c) = \tfrac{4}{5}. \quad (4.7)$$

This universal *rational* number† has been derived for the first time for the quantum nonlinear $\mathrm{O}(N)$ sigma model in the limit $N \to \infty$ [4]. It demonstrates that at the quantum critical point $\lambda = \lambda_c$ the C-function does not depend on the temperature. The difference from the

† Let us note that $\zeta(3)$ is an irrational number, as was pointed out by Apéry in 1978 (see [17]). Therefore, none of the intermediate steps suggests that a rational number will be the final result.

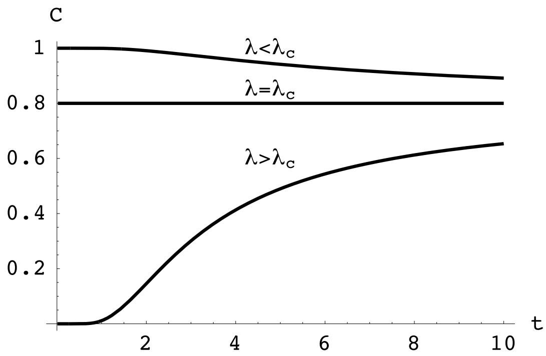

Figure 3. The behaviour of the 2D C-function is illustrated for $\lambda = 1.5\lambda_c$, $\lambda = \lambda_c$ and $\lambda = 0.5\lambda_c$.

corresponding results in [3] (cf figure 3 in [3] with figure 3 in this paper) is due to the fact that terms proportional to the difference between y and y_0 in (4.3) have been neglected there. The above is justified when $y \gg 1$ (then y and y_0 are exponentially close to each other). The analysis of the corresponding equation shows that the last happens when $x \ll -1$ (i.e. $\lambda > \lambda_c$) where $y \sim y_0 \sim (4x)^2$, which is the case of the quantum disordered region. In this case it is easy to see that

$$C(t,\lambda) \simeq \frac{4\pi}{\zeta(3)} \frac{|1-\lambda/\lambda_c|}{t} \exp[4\pi(1-\lambda/\lambda_c)t^{-1}] \tag{4.8}$$

i.e. C approaches zero exponentially fast in terms of the scaling parameter x. The behaviour of the C-function in this case is that one of massive free bosons [3].

Let us consider now the monotonicity of the C-function. From (4.3) it follows that

$$\frac{\partial C(t,\lambda)}{\partial t} = -\frac{\pi}{\zeta(3)}(y-y_0)(1-\lambda/\lambda_c)t^{-2}. \tag{4.9}$$

Since $y > y_0$, we conclude that C is a monotonically increasing function of the temperature for $\lambda > \lambda_c$, and a monotonically decreasing function for $\lambda < \lambda_c$. Within exponentially small-in-temperature corrections this result coincides, in fact, with the corresponding one for the QNLσM in the limit $N \to \infty$ [3].

The above results for the behaviour of the C-function are illustrated in figure 3.

Finally, it seems worthwhile to mention that, as follows from equation (4.3), *the C-function is a monotonically increasing function of the scaling variable x* (see figure 4) *for any value of t* ($\lambda/t \gg 1$).

4.3. The case $d = 4$

For this case, taking into account that in equation (1.1) $n(4) = 3\zeta(5)/(2\pi)^2$ and $v = \sqrt{gJ}$, for the C-function we obtain from equations (3.13) and (3.14) that $C(t,\lambda) = X(x, \lambda/t)$, where

$$X(x,\lambda/t) = \frac{(2\pi)^2}{3\zeta(5)}\left[x(y-y_0) + \frac{1}{4}r(y^2-y_0^2)\frac{\lambda}{t} + \frac{1}{(2\pi)^2}[y\mathrm{Li}_3(\exp(-\sqrt{y})) + 3\sqrt{y}\mathrm{Li}_4(\exp(-\sqrt{y})) + 3\mathrm{Li}_5(\exp(-\sqrt{y}))]\right] \tag{4.10}$$

with

$$x = \frac{1}{2}\left(\frac{\lambda}{t}\right)^3\left(\frac{1}{\lambda} - \frac{1}{\lambda_c}\right) \tag{4.11}$$

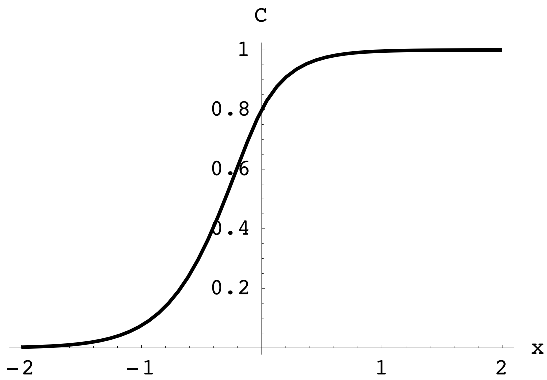

Figure 4. The behaviour of 2D C as a function of the scaling parameter $x = \pi\,(1/\lambda - 1/\lambda_c)\,\lambda/t$.

and $\lambda_c = 1/\mathcal{W}_4(0) \approx 5.2882$. Here $y \geqslant 0$, and $y_0 \geqslant 0$ are solutions of the corresponding equations that follow from (4.10) by requiring that the first partial derivative of the rhs of (4.10) with respect to y, and y_0, respectively, to be zero. This leads to the following equation for y:

$$x = -\frac{1}{2} r \frac{\lambda}{t} y + \frac{1}{2(2\pi)^2}[\sqrt{y}\mathrm{Li}_2(\exp(-\sqrt{y})) + \mathrm{Li}_3(\exp(-\sqrt{y}))]. \tag{4.12}$$

It is easy to see that, for a given t and λ, the solution of the above equation, if it exists, is unique. For y_0 we get

$$y_0 = \begin{cases} -\tilde{x} = -(2t)/(r\lambda)x & \lambda > \lambda_c \\ 0 & \lambda \leqslant \lambda_c. \end{cases} \tag{4.13}$$

One observes that in the most general case the function X, given by equation (4.10), could not be recast in a scaling form. However, as we will see below, the last is possible in some subregions of the λ–t plane. We recall that the susceptibility χ of the system is proportional to y^{-1} (if $y \neq 0$ [18]†), which leads to the conclusion that a nonzero-temperature phase transition exists at a given $t_c = t_c(\lambda)$, where $t_c(\lambda)$ is given by the equation

$$t_c(\lambda) = \lambda \left[\frac{(2\pi)^2}{\zeta(3)}\left(\frac{1}{\lambda} - \frac{1}{\lambda_c}\right)\right]^{1/3} \tag{4.14}$$

(at $t = t_c(\lambda)$ one has $y = 0$, and $y = 0$ also for $t < t_c(\lambda)$). As for the $d = 2$ case three principal different regimes exist: (i) *renormalized classical* (where y tends to zero exponentially fast as a function of λ/t); (ii) *quantum critical* (where y tends to zero algebraically as a function of λ/t or $y = \mathrm{O}(1)$); (iii) *quantum disordered* (where $y \gg 1$). In order to describe the behaviour of the C-function below we analyse these three regimes.

(A). Let us first suppose that $y \ll 1$. Then equation (4.12) becomes

$$\left(\frac{\lambda}{t}\right)^3 \left[\frac{1}{\lambda} - \frac{1}{\lambda_c} - \frac{\zeta(3)}{(2\pi)^2}\left(\frac{t}{\lambda}\right)^3\right] = \frac{1}{(4\pi)^2} y \ln(y/e) - r\frac{\lambda}{t} y. \tag{4.15}$$

† If $y = 0$ the relation between the susceptibility and y is a bit more subtle for dimensionalities above the upper critical dimension; see, e.g. [18], ch 5.

Obviously, there are two subregimes: (a) when the first term in the right-hand side dominates and (b) when the second one dominates. The borderline between them is given by

$$\frac{1}{2r}\left(\frac{\lambda}{t}\right)^2\left[\frac{1}{\lambda_c}-\frac{1}{\lambda}+\frac{\zeta(3)}{(2\pi)^2}\left(\frac{t}{\lambda}\right)^3\right]=\exp\left[-(4\pi)^2 r\frac{\lambda}{t}+1\right]. \tag{4.16}$$

In the λ–t plane equation (4.16) determines a line $t^*(\lambda)$ that is exponentially close to (as a function of λ/t) the line $t_c(\lambda)$. At $t^*(\lambda)$ the solution of equation (4.15) is

$$y\sim\exp\left[-(4\pi)^2 r\frac{\lambda}{t}\right] \tag{4.17}$$

whereas $y = 0$ at $t_c(\lambda)$. We conclude that the *renormalized classical regime* is observed for parameters of the system lying in the λ–t plane between $t_c(\lambda)$ and $t^*(\lambda)$. In this regime one could neglect the second term in the rhs of (4.10) which leads to a scaling form of the C-function with a scaling variable x, defined in (4.11). At $t^*(\lambda)$ the C-function could not be rewritten in a scaling form. In the remainder we will see that another scaling variable $\tilde{x}$ can be defined for the region to the right of $t^*(\lambda)$. This is due to the fact that in this region the first term in the rhs of equation (4.12) is of the leading order. Indeed, this is true not only for case (b) but also for cases (B), when $y = O(1)$, and (C), when $y \gg 1$, which cases are to be considered below.

Before passing to the consideration of case (B) let us note that further inspection of equation (4.15) for case (b) leads to the conclusion that there exists a crossover line

$$t_s(\lambda)=\lambda\left[\frac{(2\pi)^2}{\zeta(3)}\left(\frac{1}{\lambda_c}-\frac{1}{\lambda}\right)\right]^{1/3} \tag{4.18}$$

between two regimes where $\chi(t,\lambda)\sim t^{-3}$ and $\chi(t,\lambda)\sim t^{-2}$, respectively. This curve is symmetric to the curve $t_c(\lambda)$ with respect to the line $\lambda=\lambda_c$. Let us turn now to case (B).

(B) $y = O(1)$. Then, since $t \ll 1$, equation (4.12) becomes extremely simple and, up to the leading order coincides with the corresponding equation for the zero-temperature system (see equation (4.13)). Its solution is

$$y=\frac{1}{r}\left(\frac{\lambda}{t}\right)^2\left(\frac{1}{\lambda_c}-\frac{1}{\lambda}\right)\equiv-\tilde{x}. \tag{4.19}$$

Since $\chi(t,\lambda)\sim(yt^2)^{-1}$ and $y = O(1)$, one concludes that in this regime $\chi(t,\lambda)\sim t^{-2}$. In a given sense a formal curve $t_1(\lambda)$ in the λ–t plane which borders the region in which $y = O(1)$ can be obtained by simply setting $y = 1$ in equation (4.19). Summarizing the results from (A) and (B) we are led to the conclusion that the *quantum critical regime* is observed for values of the parameters t and λ lying in the λ–t plane between the curves $t^*(\lambda)$ and $t_1(\lambda)$. We see also that in this region $X(t,\lambda) = X(\tilde{x})$, i.e. the scaling is restored with the new scaling variable $\tilde{x}$. We pass now to case (C).

(C) $y \gg 1$. Then one formally receives the same solution as given by equation (4.19) but now $\chi(t,\lambda)\sim(1/\lambda_c-1/\lambda)^{-1}$, i.e. it does not depend on t up to exponentially small in λ/t corrections. We conclude that the region of parameters in the λ–t plane below $t_1(\lambda)$ determines the *quantum disordered* region. Again, as in (B), the scaling variable of C is $\tilde{x}$.

The above results are summarized in figure 5.

The existence of the regions of thermodynamic parameters determined above is reflected by the corresponding behaviour of the C-function given by equation (4.10).

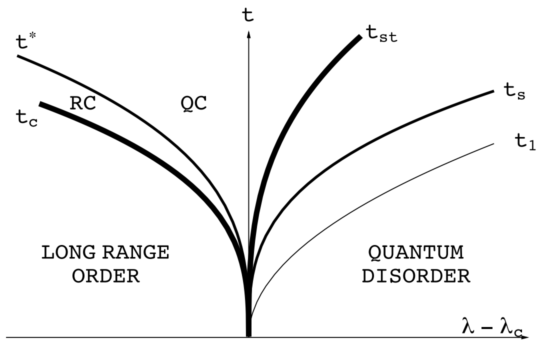

Figure 5. The phase diagram of the model and crossovers for the case $d = 4$. Long-range order exists below the curve t_c. The curve t_{st} is the locus of points in the thermodynamic space where $\partial C/\partial t = 0$. The other curves denote crossovers between different regimes which are described in the text.

First, for $t \leqslant t_c(\lambda)$ (i.e. under the existence of a long-range order in the system), since $y = y_0 = 0$, one immediately obtains from (4.10) that $C = 1$†.

Further, for $t > t_c(\lambda)$ taking into account equations (4.13) and (4.12), it is easy to see that

$$\frac{\partial C(t,\lambda)}{\partial t} = -\frac{2\pi^2 r}{\zeta(5)}\frac{\lambda}{t^2}(y - y_0)\left[\tilde{x} + \frac{1}{6}r(y + y_0)\right]. \tag{4.20}$$

Since $y > y_0 > 0$ for the considered region of parameters λ and t, the above equation leads us to the conclusion that for any $\lambda < \lambda_c$ the C-function is a *monotonically decreasing function of the temperature*. The same is true also for $\lambda = \lambda_c$ and $t \neq 0$. From the analysis of the solutions of the equations for y and y_0 it becomes clear that for a fixed λ and $t > t_s(\lambda)$ the *leading-order* form for both y and y_0 is $y = y_0 = -\tilde{x}$ (if one takes into account next-to-leading order terms then, of course, $y > y_0$). Setting the above expressions for y and y_0 in the rectangular brackets in equation (4.20), we conclude that C is a *monotonically increasing function of* t for any $t > t_s(\lambda)$. It is clear that somewhere between $\lambda = \lambda_c$ and $t_s(\lambda)$ the derivative of the C-function changes its sign, i.e. there is a line $t_{st}(\lambda)$ of stationary points $\partial C(t,\lambda)/\partial t = 0$. One can see that

$$t_{st}(\lambda) = \lambda\left[\frac{(4\pi)^2}{\zeta(3)}\left(\frac{1}{\lambda_c} - \frac{1}{\lambda}\right)\right]^{1/3}. \tag{4.21}$$

Since the point $\lambda = \lambda_c, t = 0$ lies on $t_{st}(\lambda)$ and at it $C = 1$, we conclude that $C = 1$ at the whole line $t_{st}(\lambda)$. It is clear now that C is a *monotonically increasing* function of t for $t < t_{st}(\lambda)$. It is a *monotonically decreasing* function of t for $t > t_{st}(\lambda)$ as well as for any (small) t if $\lambda < \lambda_c$. These results are summarized in figure 6.

5. FSS interpretation

It is interesting to interpret the bulk critical behaviour of the C-function in the context of the FSS theory by introducing a finite 'temporal' dimension $L_\tau = \lambda/t$. Then, taking into account: (i)

† Note that this result depends only on the existence of long-range order in the system (then $y = y_0 = 0$) and not on the dimensionality d. From equations (1.1) and (3.3)–(3.5) and the identity

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \int_0^\infty R(\pi s)x^{s/2}\frac{\mathrm{d}x}{x} \qquad \mathrm{Re}\, s > 1$$

we get $C = 1$.

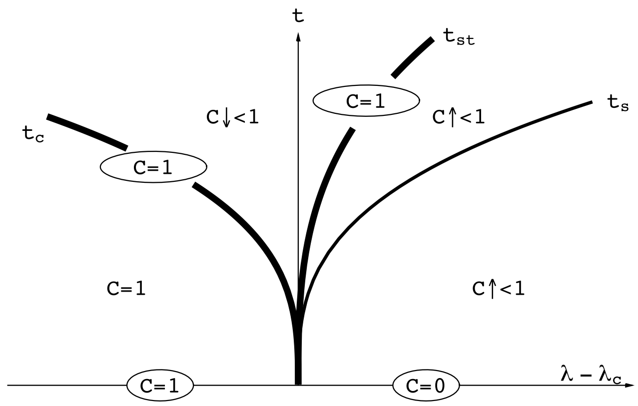

Figure 6. The monotonic behaviour of C as a function of temperature is shown. The symbols $C\uparrow$ ($C\downarrow$) mean that C is a monotonically increasing (decreasing) function of the temperature. The number in ellipses show the value of C at the corresponding curve. In whole long-range-order region, i.e. below the line t_c, $C = 1$.

the dimensional crossover rule that connects the properties of a given d-dimensional quantum system and those ones of the corresponding $(d+1)$-dimensional classical one with the mapping $d \to \mathrm{d}+1$, $L \to L_\tau$, $t \to \lambda$; (ii) the Privman and Fisher [19] hypothesis for the free energy of a finite classical system when the hyperscaling holds (i.e. between the lower d_l and the upper d_u critical dimensions of the system), one could make the statement that the free energy f_∞ of a quantum system with dimensionality $d_l < d < d_u$ (for our model $d_l = 1$ and $d_u = 3$) should have the form

$$f_\infty(t,\lambda|d) - f_\infty(0,\lambda|d) = T L_\tau^{-d} Y\left(\frac{L_\tau}{\xi(0,\lambda)}\right) = T^{1+d} v^{-d} Y\left(\frac{v}{T\xi(0,\lambda)}\right) \qquad (5.1)$$

where $\xi(0,\lambda)$ is the correlation length of the zero-temperature system, Y is a *universal function* and $v = TL_\tau$. As mentioned in the introduction, one interprets v as a characteristic velocity in the system. Recall that in order to have no nonuniversal prefactor in front of Y for classical systems one considers $\tilde{f}_\infty = \beta f_\infty$, instead of f_∞ itself. The normalization of the free energy in (5.1) simply follows from our choice of L_τ, or, equivalently, v in the system. For the model considered here the inspection of equations (4.3) and (5.1) shows that the hypothesis (5.1) is indeed valid with the standard scaling variable $L_\tau/\xi(0,\lambda) \equiv x = \pi(1/\lambda - 1/\lambda_c)\lambda/t$, $C(t,\lambda) = X(x) = -Y(x)/n(d)$, and $v = \sqrt{gJ}$. It is interesting that despite the lack of hyperscaling at $d_l = 1$ the C-function again can be written as a function of $L_\tau/\xi(0,\lambda)$ (see equation (4.1)), if one identifies $\xi(0,\lambda) = \phi_0^{-1/2}$, where ϕ_0 is given by equation (4.2). The case $d = 4 > d_u$ is much more interesting due to the lack of hyperscaling. In the most general case the C-function could not even be recast in a FSS form (see equation (4.10)). The latter is possible exponentially close (in L_τ) to the line of finite-temperature phase transitions $t_c(\lambda)$, where the modified scaling variable is $2x = L_\tau[L_\tau/\xi(0,\lambda)]^2$ (see equation (4.11)). The standard scaling variable $\tilde{x} = L_\tau/\xi(0,\lambda)$ is restored only for parameters to the right of the curve $t^*(\lambda)$ in the λ–t plane (see equation (4.19) and the comments connected with it). This change of scaling variables from x to $\tilde{x}$ is a new point within FSS theory. Normally one observes modified FSS (see [20, 21]) above d_u due to the existence of dangerous irrelevant variables in the system. On the other hand, considering the 5D spherical model with one finite dimension, Barber and Fisher, as early as 1973 [22], stated that the scaling variable should be the standard one, i.e. $L_\tau/\xi(0,\lambda)$ in our notation. The above results resolve this seeming contradiction: the scaling variable has to be modified very close to the phase boundary, but is the standard one a bit away from it. The physical reasoning for that difference is the existence in the system of a temperature-driven phase transition in addition to the quantum one with respect to λ at $t = 0$. To our knowledge all other examples considered previously in the literature of modified FSS concern finite systems with no (sharp) phase transition in them.

6. Concluding remarks

One generally expects that the C-function increases monotonically when the quantum fluctuations 'dominate' [3]. The real meaning of the term 'dominate' turns out to be quite subtle, as we have demonstrated in the current paper. In fact, we have shown that the region where the C-function remains monotonically increasing (as a function of temperature) and the quantum critical region do essentially intersect but do *not* coincide (see figure 2). This is one of the results of the present work. The question of where one should look and what should be understood as the domination of quantum fluctuations is, indeed, very intriguing. It is a part of the more general problem of a *quantitative* description of the interplay of the quantum and critical fluctuations. There exist different views on that issue. The standard one [23, 24] is based on the 'ratio' between the correlation length and the length of de Broglie. Another possible approach can be based on the behaviour of the C-function [3,5]. Furthermore, there is an approach based on the algebra of critical fluctuation operators, due to Verbeure and Zagrebnov [25], where a measure of the 'degree of criticality' is introduced in a mathematically rigorous way.

In the present work we investigated the behaviour of the C-function for $d = 1, 2, 4$. The case $d = 1$ represents the situation with no phase transition and strong quantum fluctuations, $d = 2$—the one when a quantum critical point appears at $T = 0$, and $d = 4$—when there is a line of classical critical points ending up with a zero-temperature (quantum) critical point. In fact, these are the most typical cases on which the attention in the literature is focused.

Case $d = 1$. As is to be expected on general grounds, the C-function increases monotonically as a function of temperature (see figure 1). This reflects the fact that the quantum fluctuations are strong enough (as is clear from equation (4.2), one cannot consider λ as a small parameter) and the lack of a critical point. The C-function obtained here coincides with the C-function of the massive free bosons (for $d = 1$) with mass $\sqrt{\phi_0}$, because of the exponentially small difference then between ϕ and ϕ_0, i.e. one can consider ϕ as a fixed parameter in (3.8) and in (3.9). The general case (for any d) of free massive bosons actually follows from (3.3)–(3.6) by considering ϕ there as a fixed parameter connected to the mass m of bosons ($\phi \sim m^2$). For the last case it is trivial to check that the corresponding C-function is that obtained in [3] (see equation (3.2) there).

Case $d = 2$. Figure 2 shows the phase diagram for our model which coincides with the phase diagram of the $d = 1$ quantum Ising model, as well as with the nonlinear O(n) sigma model in the limit $n \to \infty$: see, e.g., Sachdev [23]. As a function of the temperature, C is monotonically increasing for λ above λ_c, equals $\frac{4}{5}$ at $\lambda = \lambda_c$ (and then C does not depend on t) and is monotonically decreasing for λ below λ_c (see figure 3). The lack of overall monotonicity with respect to the temperature is due to the crossover from classical to quantum behaviour.

It is clear, that one can indeed consider the *monotonicity of C as a measure* of the role the corresponding fluctuations are playing in a given region of parameters. It is interesting that C changes its monotonicity, in fact, in the *middle* of the quantum critical region. Finally, we note that it is nevertheless possible to find a (nontrivial) variable, with respect to which the C-function is monotonic in the whole t–λ plane (see figure 4). This variable is the scaling variable $x = \pi(1/\lambda - 1/\lambda_c)\lambda/t$.

Case $d = 4$. The existence of a line of nonzero temperature critical points modifies drastically the corresponding picture in comparison with the $d = 2$ case. Now a line of stationary points $t_{st}(\lambda)$ appears (see figure 5) which 'starts' from $(\lambda = \lambda_c, t = 0)$ and lies to the left of $t_s(\lambda)$. To the left of $t_{st}(\lambda)$, C is a nonincreasing function of the temperature (see figure 6). For $\lambda < \lambda_c$ and $t < t_c(\lambda)$ one has $C = 1$, whereas within the region between $t_c(\lambda)$ and $t_{st}(\lambda)$ the C-function is monotonically decreasing as a function of the temperature. To the right of $t_{st}(\lambda)$ the C-function becomes a monotonically increasing function of the temperature, being zero at the $t = 0$, $\lambda > \lambda_c$ line. At the lines $t_c(\lambda)$ and $t_{st}(\lambda)$ the C-function reaches its maximum value, i.e. it becomes $C = 1$. Finally, we would like to mention that, similar to the case $d = 2$, it is possible to find two nontrivial parameters such that with respect to both of them the C-function is monotonically increasing. Such parameters are, e.g., the parameters x and λ/t (see equation (4.10) and take into account that $y > y_0$).

Comparing the behaviour of the C-function for $d = 1$, $d = 2$ and $d = 4$ we conclude that:

(a) For $d = 1$ for any fixed λ we have a C-function that is monotonically increasing with temperature.
(b) For $d = 2$ the above is true only for $\lambda > \lambda_c$.
(c) For $d = 4$ the C-function is a monotonically increasing function of t for $\lambda > \lambda_c$ and t *small enough*. The monotonicity of the C-function does not change by increasing t *only* for $\lambda = \lambda_c$.

So, the region in the parametric space where C remains monotonically increasing with t becomes smaller when d increases. We explicitly see the crucial role the dimensionality d and the existence of phase transition, which appears upon increasing d, play in the behaviour of the C-function as a function of t. Nevertheless, for any d one can find nontrivial variable(s), function(s) of the temperature and the parameter controlling the quantum fluctuations, in terms of which C is a monotonically increasing function of its variable(s). In close vicinity of the quantum critical point the C-function is given by a universal scaling function whose properties can be interpreted in terms of FSS which has to be modified for $d = 4$.

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